

ON THE ISOMETRY GROUP OF $RCD^*(K, N)$ -SPACES.

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ABSTRACT. We prove that the group of isometries of a metric measure space that satisfies the Riemannian curvature condition, $RCD^*(K, N)$, is in fact a Lie group. We obtain an optimal upper bound on its dimension and classify the spaces where this maximal dimension is achieved.

1. INTRODUCTION.

The notion of synthetic lower Ricci curvature bounds was independently defined for metric measure spaces by Sturm ([24], [25]) and by Lott-Villani (see for example [26]). This condition, called curvature-dimension condition $CD(K, N)$, is given in terms of the optimal transport between probability measures on X and the convexity of an entropy functional. It is known that a Riemannian manifold (M, g) having Ricci curvature bounded below is equivalent to the metric measure space $(M, d_g, dvol_g)$ being a $CD(K, N)$ space.

We say that a metric measure space (X, d, \mathbf{m}) satisfies the Riemannian curvature dimension condition, also called *infinitesimally Hilbertian*, $RCD(K, N)$, if the associated Sobolev space $W^{1,2}$ is in fact a Hilbert space. This condition rules out Finsler geometries and provides good structural results such as an extension of the classical Splitting theorem of Cheeger-Gromoll done by Gigli [10]. In fact, this last result does not extend to the larger class of $CD(K, N)$ spaces as one can see by considering the case of $(\mathbb{R}^n, d_{\|\cdot\|}, \mathcal{L}^n)$, where $d_{\|\cdot\|}$ is a distance induced by a norm and \mathcal{L}^n is the Lebesgue measure. If the norm does not come from a interior product then the space is $CD(0, n)$, has plenty of lines, but does not split. For details of this we refer the reader to [26].

In [2] Bacher-Sturm defined via some distortion coefficients a curvature dimension condition that has better local-to-global and tensorization properties. This condition is called reduced curvature dimension condition and is denoted by $CD^*(K, N)$. It is worth noting that every $CD(K, N)$ space is also a $CD^*(K, N)$ space, and that a $CD^*(K, N)$ space is $CD(\frac{N-1}{N}K, N)$. A space that is both $CD^*(K, N)$ and infinitesimally Hilbertian will be denoted as $RCD^*(K, N)$. Mondino-Naber [18] have developed a structure theory of $RCD^*(K, N)$ spaces that includes a stratification into subsets of points that have Euclidean tangent spaces. Later Gigli-Mondino-Rajala [11] proved that, in fact, the tangent space is unique almost everywhere and it is Euclidean. However the dimension of such tangents might vary point-wise.

Isometric actions on Riemannian manifolds have been a useful tool to investigate the interaction between the topology and the Riemannian metric a manifold might admit. They also provide a good link between theory and examples. A major result in this area is the theorem of Myers-Steenrod [21] that proves that the isometry group of a Riemannian manifold is in fact a Lie group. A good reference for this and similar results is [14].

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The aim of this paper is to study the isometry group of $RCD^*(K, N)$ -spaces. The first, necessary step is to obtain an analogue of the Myers-Steenrod Theorem to $RCD^*(K, N)$ -spaces, as appears in the following statement.

Theorem 1. *Let (X, d_X, \mathbf{m}) be a $RCD^*(K, N)$ -space with $K \in \mathbb{R}$, $N \in [1, \infty)$. Then $\text{Iso}(X)$, the isometry group of X , is a Lie group.*

The proof is obtained by extending to our situation the corresponding statement for Alexandrov spaces [8]. Although most of our proof is similar to theirs, there were parts where the lack of the extra structure that Alexandrov spaces enjoy, forced us to providing several details that were missing in [8]. Once this is accomplished, we bound the dimension of the isometry group in terms of N .

Theorem 2. *Let (X, d, \mathbf{m}) be a $RCD^*(K, N)$ space with $K, N \in \mathbb{R}$ and $N < \infty$, then the Hausdorff dimension of G , the isometry group of X , is bounded above by $\frac{[N]([N]+1)}{2}$.*

Finally, we describe the spaces whose isometry group attains the maximal dimension allowed. In order to do this, we show that such spaces are necessarily smooth manifolds, which allows us to use well known results of Kobayashi in [14].

Theorem 3. *Let (X, d, \mathbf{m}) be a connected $RCD^*(K, N)$ space of topological dimension $[N]$. If the dimension of the isometry group of X is $[N]([N] + 1)/2$, then X is isometric to one of the following space forms:*

- (1) An $[N]$ -dimensional Euclidean space $\mathbb{R}^{[N]}$.
- (2) An $[N]$ -dimensional sphere $\mathbb{S}^{[N]}$.
- (3) An $[N]$ -dimensional projective space $\mathbb{R}P^{[N]}$.
- (4) An $[N]$ -dimensional simply connected hyperbolic space $\mathbb{H}^{[N]}$.

The paper is organized as follows: section 2 gives preliminaries and general facts needed for the proofs of the rest of Theorems in the paper; section 3 proves that the isometry group of an $RCD^*(K, N)$ -space is a Lie group; in section 4, we bound the maximal dimension of the isometry group of an $RCD^*(K, N)$, and finally, section 5 deals with the case of the rigidity that appears for the space when the dimension of its isometry group is maximal.

2. PRELIMINARIES.

Given two metric measure spaces (X, d_X, μ) and (Y, d_Y, ν) , we say that they are *isomorphic* if there exists an isometry $T : (\text{supp}(\mu), d_X) \rightarrow (\text{supp}(\nu), d_Y)$ such that $T_{\#}\mu = \nu$. Observe that every metric measure space (X, d_X, μ) is isomorphic to $(\text{supp}(\mu), d_X, \mu)$ so from this point we will assume that $\text{supp}(\mu) = X$. We will denote by $\mathbb{P}_2(X)$ the space of probability measures on X with finite second moments, that is

$$\int_X d^2(x_0, x) d\mu(x) < \infty,$$

for some (and hence for all) $x_0 \in X$.

Let $\mu_0, \mu_1 \in \mathbb{P}_2(X)$. A measure $\pi \in \mathbb{P}(X \times X)$ is a coupling between μ_0 and μ_1 if for the projections $\text{proj}_i : X \times X \rightarrow X$, $i = 1, 2$ we have $\text{proj}_{1\#} \pi = \mu_0$, $\text{proj}_{2\#} \pi = \mu_1$. We define the L^2 -Wasserstein distance between μ_0 and μ_1 as:

$$\mathbb{W}_2^2(\mu_0, \mu_1) := \inf \left\{ \int_{X \times X} d(x, y)^2 d\pi(x, y) \mid \pi \text{ is a coupling of } \mu_0, \mu_1 \right\}.$$

A measure that satisfies the infimum will be called an optimal coupling between μ_0 and μ_1 .

A curve $\gamma : [0, l] \rightarrow X$ is a geodesic if its length coincides with the distance between its endpoints. We will call a metric space (X, d) a *geodesic space* if for any two points in X there exists a geodesic that connects them.

It is known that if (X, d) is a Polish geodesic space then the L^2 -Wasserstein space $(\mathbb{P}_2(X), \mathbb{W}_2)$ is also Polish and geodesic (see for example: [24], [26]). Let $Geo(X)$ denote the set of constant speed geodesics from $[0, 1]$ to (X, d) equipped with the *sup* norm.

We can lift any geodesic $(\mu_t)_{t \in [0, 1]} \in Geo(\mathbb{P}_2(X))$ to a measure $\pi \in \mathbb{P}(Geo(X))$, in such way that $(e_t)_\# \pi = \mu_t$, where $e_t : Geo(X) \rightarrow X$ is the evaluation map at $t \in [0, 1]$. We will then denote by $OptGeo(\mu_0, \mu_1)$ the space of probability measures $\pi \in \mathbb{P}(Geo(X))$ such that $(e_0, e_1)_\# \pi$ is an optimal coupling between μ_0 and μ_1 .

We will say that an optimal coupling $\pi \in \mathbb{P}(Geo(X))$ is essentially non-branching if it is concentrated on a set of non-branching geodesics.

Definition 2.1. Let (X, d_X, p_X) , (Y, d_Y, p_Y) be Polish metric spaces. For $\epsilon > 0$ we will call a function $f_\epsilon : B_{p_X}(1/\epsilon, X) \rightarrow B_{p_Y}(1/\epsilon, Y)$ a *pointed ϵ -Gromov-Hausdorff approximation* if:

- (1) $f_\epsilon(p_X) = p_Y$,
- (2) For all $u, v \in B_{p_X}(1/\epsilon, X)$, $|d_X(u, v) - d_Y(f_\epsilon(u), f_\epsilon(v))| < \epsilon$, and
- (3) For all $y \in B_{p_Y}(1/\epsilon, Y)$ there exists $x \in B_{p_X}(1/\epsilon, X)$ such that $d_Y(f_\epsilon(x), y) < \epsilon$.

Note that we do not assume that f_ϵ is continuous.

The pointed Gromov Hausdorff distance between two pointed metric spaces (X, d_X, p_X) , (Y, d_Y, p_Y) , denoted by $d_{GH}((X, d_X, p_X), (Y, d_Y, p_Y))$, is by definition equal to the infimum of all ϵ such that there exists a ϵ -Grommet-Hausdorff approximation from (X, d_X, p_X) to (Y, d_Y, p_Y) , and from (Y, d_Y, p_Y) to (X, d_X, p_X) .

Definition 2.2 (measured Gromov-Hausdorff convergence). Let $(X_k, d_k, \mathbf{m}_k, p_k)_{k \in \mathbb{N}}$ and $(X_\infty, d_\infty, \mathbf{m}_\infty, p_\infty)$ be pointed metric measure spaces. We will say that X_k converges to X_∞ in the pointed measured Gromov-Hausdorff topology if there exist Borel measurable ϵ_k -Gromov-Hausdorff approximations such that $\epsilon_k \rightarrow 0$ and

$$(f_{\epsilon_k})_\# \mathbf{m}_k \rightarrow \mathbf{m}_\infty$$

weakly.

Let (X, d, \mathbf{m}) be a metric measure space, $x_0 \in X$ and $r > 0$; we consider the rescaled and normalized pointed metric measure space $(X, d_{rX}, \mathbf{m}_r, x_0)$ where $d_{rX}(x, y) := rd(x, y)$ for all $x, y \in X$ and \mathbf{m}_r is given by

$$(1) \quad \mathbf{m}_r := \left(\int_{cl(B_{x_0}(1/r, X))} 1 - d_{rX}(\cdot, x_0) d\mathbf{m} \right)^{-1} \mathbf{m}.$$

Sometimes we will use the notation $\mathbf{m}_r^{x_0}$ if we want to stress the dependence on x_0 .

Definition 2.3 (Tangent cone). Let (X, d, \mathbf{m}) be a metric measure space and $x_0 \in X$. A pointed metric measure space $(Y, d_Y, \mathbf{n}, y_0)$ is called a *tangent cone of X at x_0* if there exists a sequence $(r_i) \subset \mathbb{R}$, $r_i \rightarrow \infty$ so that $(X, d_{r_i X}, \mathbf{m}_i, x_0)$ converges to $(Y, d_Y, \mathbf{n}, y_0)$ in the pointed measured Gromov-Hausdorff topology. We denote the collection of all tangent cones of X at x_0 by $Tan(X, x_0)$.

It is important to notice that Tangent cones, unlike their analogues in Riemannian manifolds or Alexandrov spaces, may depend on the sequence one considers, examples of this behaviour are studied in [6].

First we define the distortion coefficients $\sigma_{K,N}^{(t)}(\theta)$ for $K \in \mathbb{R}$ and $N \in [1, \infty)$:

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \geq N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\ t & \text{if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 < 0. \end{cases}$$

Definition 2.4 (*CD^{*}(K, N) curvature condition.*). Let $K \in \mathbb{R}$ and $N \in [1, \infty)$. A metric measure space (X, d, \mathbf{m}) is said to be a $CD^*(K, N)$ space if for any two measures $\mu_0, \mu_1 \in \mathbb{P}_2(X)$ with bounded support contained in $\text{supp}(\mathbf{m})$ and with $\mu_0, \mu_1 \ll \mathbf{m}$, there exists an optimal coupling π such that for every $t \in [0, 1]$ and $N' \geq N$ one has,

$$-\int \rho_t^{1-\frac{1}{N'}} d\mathbf{m} \leq -\int \sigma_{K,N'}^{(1-t)}(d(\gamma_0, \gamma_1)) \rho_0^{-\frac{1}{N'}} + \sigma_{K,N'}^{(t)}(d(\gamma_0, \gamma_1)) \rho_1^{-\frac{1}{N'}} d\pi(\gamma),$$

where ρ_t , $t \in [0, 1]$, is the Radon-Nikodym derivative $d(e_t)_{\#}\pi/d\mathbf{m}$.

Let $Lip(X)$ denote the set of Lipschitz functions in X . For every $f \in Lip(X)$, the local Lipschitz constant at x , $|Df|(x)$, is defined by

$$|Df|(x) := \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)},$$

when x is not isolated, otherwise $|Df|(x) := \infty$.

Definition 2.5 (*Cheeger energy*). Let $f \in L^2(X, \mathbf{m})$, we define the Cheeger energy of f as

$$Ch(f) := \frac{1}{2} \inf \left\{ \liminf_{n \rightarrow \infty} \int_X |Df_n|^2 d\mathbf{m}; f_n \in Lip(X), f_n \rightarrow f \text{ in } L^2 \right\}.$$

Set $D(Ch) := \{f \in L^2(X, \mathbf{m}); Ch(f) < \infty\}$.

We say that (X, d, \mathbf{m}) is infinitesimally Hilbertian if the Cheeger energy is a quadratic form. It is equivalent to the Sobolev space $W^{1,2}(X, d, \mathbf{m}) := \{f \in L^2 \cap D(Ch)\}$ equipped with the norm $\|f\|_{1,2}^2 := \|f\|_2^2 + 2Ch(f)$ being a Hilbert space.

If a $CD^*(K, N)$ space is infinitesimally Hilbertian then it is called a $RCD^*(K, N)$ space.

To finish this section we enunciate some structural results about $RCD^*(K, N)$ spaces on which our arguments rely heavily upon.

Theorem 2.6 (*Hausdorff dimension* [25]). Let (X, d, \mathbf{m}) be a $CD^*(K, N)$ space, with $K \in \mathbb{R}$ and $N \geq 1$. Then X has Hausdorff dimension bounded above by N .

Theorem 2.7 (*Splitting* [10]). Let (X, d, \mathbf{m}) be a $RCD^*(0, N)$ -space with $1 \leq N \leq \infty$. Suppose that $\text{supp}(\mathbf{m})$ contains a line. Then (X, d, \mathbf{m}) is isomorphic to $(X' \times \mathbb{R}, d' \times d_E, \mathbf{m}' \times \mathcal{L}^1)$, where d_E is the Euclidean metric, \mathcal{L}^1 is the Lebesgue measure and (X', d', \mathbf{m}') is a $RCD^*(0, N-1)$ -space if $N \geq 2$ and a singleton if $N < 2$.

Theorem 2.8 (*Stratification* [18]). Let (X, d, \mathbf{m}) be a $RCD^*(K, N)$ -space with $1 \leq N \leq \infty$, $K \in \mathbb{R}$. Define for $1 \leq k \leq [N]$, $\mathcal{R}_k := \{x \in X | \mathbb{R}^k \in \text{Tan}(X, x) \text{ but } \mathbb{R}^{k+j} \notin \text{Tan}(X, x) \text{ for every } j \geq 1\}$. Then

- (1) Every \mathcal{R}_k is \mathbf{m} -measurable and $\mathbf{m}\left(X - \bigcup_{1 \leq k \leq [N]} \mathcal{R}_k\right) = 0$.
- (2) For \mathbf{m} -a.e. $x \in \mathcal{R}_k$ the tangent cone of X at x is unique and isomorphic to the k -dimensional Euclidean space.

3. THE ISOMETRY GROUP.

The proof follows closely the corresponding proof for Alexandrov spaces given in [8]; there are, however, some necessary changes to adapt to the lack of structure of $RCD(N, K)$ -spaces versus Alexandrov spaces. We have also corrected several typos from that reference, and given some of the details that were skipped in [8].

3.1. Point picking. The main aim of this section is to prove Theorem 3.2, stating that after a careful choice of the base points, any rescaling sequence of X approaches an Euclidean space.

We start with a simple but useful calculus fact.

Lemma 3.1. *Let $\{a_n\}, \{b_n\}$, sequences of positive real numbers with $\lim a_n = \lim b_n = \infty$. Then there is a sequence $\{\epsilon_n\}$ with $\epsilon_n > 0$ such that*

$$\lim \epsilon_n = 0, \quad \lim \epsilon_n a_n = \infty, \quad \epsilon_n a_n < b_n \text{ for all } n$$

Proof. Just choose

$$\epsilon_n = \min \left\{ \frac{1}{\sqrt{a_n}}, \frac{b_n}{2a_n} \right\}.$$

□

If $\{r_i\}$ is a sequence of positive numbers, we denote the normalized measure \mathbf{m}_{r_i} by \mathbf{m}_i (see (1) for the definition of \mathbf{m}_r).

Theorem 3.2. *Let (X, d, \mathbf{m}) be an $RCD^*(K, N)$ -space with $N \in [1, \infty)$. Then for every $x \in X$ and $r_i \rightarrow \infty$ there exists a sequence $x_i \rightarrow x$ in X such that*

$$(X, d_{r_i X}, \mathbf{m}_i, x_i) \rightarrow (\mathbb{R}^{m'}, d_E, \mathcal{L}^{m'}, 0).$$

We note that $1 \leq m' \leq N$ depends on the sequence (x_i) .

Proof. We note that for $r_i \neq 0$, the space $(X, d_{r_i X}, \mathbf{m}_i, x_0)$ is an $RCD^*(r_i^{-2}K, N)$ -space that will be denoted as $(r_i X, x_0)$. We observe that the measure \mathbf{m}_i is a normalization of the original measure in such a way that $\mathbf{m}_i(B_{x_0}(1, r_i X)) = 1$.

By Gromov-Hausdorff precompactness, we have that $(r_i X, x_0) \rightarrow (Y, y_0)$, where (Y, y_0) is an $RCD^*(0, N)$ space. We will denote the pointed measured Gromov-Hausdorff distance by d_{GH} .

Let $o_i := d_{GH}((r_i X, x_0), (Y, y_0))$; by convergence, $o_i \rightarrow 0$. Consider o_i -approximations

$$f_i : B_{x_0}(1/o_i, r_i X) \rightarrow B_{y_0}(1/o_i, Y), \quad h_i : B_{y_0}(1/o_i, Y) \rightarrow B_{x_0}(1/o_i, r_i X).$$

Lemma 3.1 gives us a sequence $\epsilon_i \rightarrow 0$ such that

$$\epsilon_i r_i < \frac{1}{o_i}, \quad \lim_{i \rightarrow \infty} \epsilon_i r_i = \infty.$$

For a point $y_i \in Y$ with $d_Y(y_0, y_i) = \epsilon_i r_i$, we define $z_i = h_i(y_i)$. We then take the midpoint x'_i in a minimal geodesic between x_0 and z_i . We reparametrize this geodesic as $\gamma_i : (-a_i, a_i) \rightarrow r_i X$ where $a_i = \frac{1}{2}d_{r_i X}(x_0, z_i)$. After we change the base point to x'_i we see that γ_i converges to a line in the limit space Y' . By the splitting theorem we have that Y' is isometric to an $RCD^*(0, N)$ -space $\mathbb{R} \times Y'_1$.

If Y'_1 is a singleton, we are done; so assume it is not. We will show that it has infinite diameter. Observe that $\sqrt{r_i} \rightarrow \infty$, so as above, the sequence $(X, \sqrt{r_i} d_X, \mathbf{m}_{\sqrt{r_i}}^{x_i}, x_i)$ converges to some $RCD^*(0, N)$ -space $(\mathbb{R} \times Z, d_E \times d_Z, \mathcal{L}^1 \otimes \mathbf{m}_Z, (0, z))$. On the other

hand, the sequence $(\mathbb{R} \times Z, \sqrt{r_i}(d_E \times d_Z), (\mathcal{L}^1 \otimes \mathbf{m}_Z)_{\sqrt{r_i}}^{(0,z)}, (0, z))$ converges to some space $(\mathbb{R} \times Z', d_E \times d_{Z'}, \mathcal{L}^1 \otimes \mathbf{m}_{Z'}, (0, z'))$, where Z' has diameter 0 or ∞ . Let

$$\eta_i := d_{pmGH}((\mathbb{R} \times Z, \sqrt{r_i}(d_E \times d_Z), (\mathcal{L}^1 \otimes \mathbf{m}_Z)_{\sqrt{r_i}}^{(0,z)}, (0, z)), (\mathbb{R} \times Z', d_E \times d_{Z'}, \mathcal{L}^1 \otimes \mathbf{m}_{Z'}, (0, z'))).$$

By remark 27.18 in [26], if

$$\phi_i : B_{x_i}(1/\theta_i, \sqrt{r_i}X) \rightarrow B_{(0,z)}(1/\theta_i, \mathbb{R} \times Z)$$

is a θ_i -approximation, then for $\frac{1}{\sqrt{r_i}} \min\{1/\theta_i, 1/\eta_i\} < 1/\theta_i$, the restriction of ϕ_i

$$\phi_i : B_{x_i} \left(\frac{1}{\sqrt{r_i}} \min\{1/\theta_i, 1/\eta_i\}, \sqrt{r_i}X \right) \rightarrow B_{(0,z)} \left(\frac{1}{\sqrt{r_i}} \min\{1/\theta_i, 1/\eta_i\}, \mathbb{R} \times Z \right)$$

is a $3\theta_i$ -approximation.

Now if ψ_i is an η_i -approximation between $(\mathbb{R} \times Z, \sqrt{r_i}(d_E \times d_Z), (\mathcal{L}^1 \otimes \mathbf{m}_Z)_{\sqrt{r_i}}^{(0,z)}, (0, z))$ and $(\mathbb{R} \times Z', d_E \times d_{Z'}, \mathcal{L}^1 \otimes \mathbf{m}_{Z'}, (0, z'))$, we can take the composition $\psi_i \circ \phi_i$ to obtain a $3\theta_i + \eta_i$ -approximation.

Finally, it is clear that $B_{x_i} \left(\frac{1}{\sqrt{r_i}} \min\{1/\theta_i, 1/\eta_i\}, \sqrt{r_i}X \right) = B_{x_i}(\min\{1/\theta_i, 1/\eta_i\}, r_iX)$. Therefore

$$d_{pmGH}((X, r_i d_X, \mathbf{m}_{r_i}^{x_i}, x_i), (\mathbb{R} \times Z', d_E \times d_{Z'}, \mathcal{L}^1 \otimes \mathbf{m}_{Z'}, (0, z'))) \rightarrow 0$$

and therefore Z' is isometric to Y'_1 .

We will now refine the choice of the x'_i 's to split iteratively further lines in the limit.

Let $o'_i := d_{GH}((r_iX, x'_i), (\mathbb{R} \times Y'_1, (0, y'_0)))$; choose $\epsilon'_i \rightarrow 0$ such that

$$\epsilon'_i r_i < \min \left\{ \frac{1}{o_i}, \frac{1}{o'_i} \right\}, \quad \lim_{i \rightarrow \infty} \epsilon'_i r_i = \infty.$$

Now we take $(0, y'_i) \in \{0\} \times Y'_1$ such that $d_{Y'_1}(y'_0, y'_i) = \epsilon'_i r_i$. Let $y''_i \in Y'_1$ be the midpoint in a geodesic between y'_0 and y'_i . In $\mathbb{R} \times Y'_i$ we consider two curves: the line $\gamma'_i(s) = (s, y''_i)$, and the segment $\ell_i : [-\frac{\epsilon'_i}{2}r_i, \frac{\epsilon'_i}{2}r_i] \rightarrow \mathbb{R} \times Y'_i$ between $(0, y'_0)$ and $(0, y'_i)$. Observe that $\gamma'_i(0) = \ell_i(0) = (0, y''_i)$.

Let $h'_i : B_{(0,y'_0)}(1/o'_i, \mathbb{R} \times Y'_1) \rightarrow B_{x'_i}(1/o'_i, r_iX)$ be some o'_i -Gromov-Hausdorff approximation, and define $\tilde{\ell}_i := h'_i(\ell_i)$, and $\tilde{\gamma}'_i := h'_i(\gamma'_i)$. Use x''_i to denote $\tilde{\ell}_i(0) = \tilde{\gamma}'_i(0)$. Observe that, because of the definition of Gromov-Hausdorff approximation, and the choices of γ'_i and ℓ_i , we have that

$$\begin{aligned} (2) \quad & |d_{r_iX}(\tilde{\ell}_i(t), \tilde{\ell}_i(s)) - |t - s|| < o'_i, \\ & |d_{r_iX}(\tilde{\gamma}'_i(t), \tilde{\gamma}'_i(s)) - |t - s|| < o'_i, \\ & |d_{r_iX}(\tilde{\ell}_i(t), \tilde{\gamma}'_i(s)) - (t^2 + s^2)^{\frac{1}{2}}| < o'_i \end{aligned}$$

for all $s, t \in [-\frac{\epsilon'_i}{2}r_i, \frac{\epsilon'_i}{2}r_i]$.

If we change the base points to x''_i , the sequence (r_iX, x''_i) converges to a probably different $RCD^*(0, N)$ -space $(Y_2, d_{Y_2}, \nu_2, y''_0)$; once again, let o''_i be the Gromov-Hausdorff distance between r_iX and Y_2 , and take a corresponding o''_i -Gromov-Hausdorff approximation $f''_i : B_{x''_i}(\frac{1}{o''_i}, r_iX) \rightarrow B_{y''_0}(\frac{1}{o''_i}, Y_2)$.

Now consider $\hat{\ell}_i := f_i''(\tilde{\ell}_i)$, $\hat{\gamma}'_i := f_i''(\tilde{\gamma}'_i)$. For a fixed s, t , and i large enough, we get from (2) that

$$(3) \quad \begin{aligned} |d_{Y_2}(\hat{\ell}_i(t), \hat{\ell}_i(s) - |t - s|| &< o'_i + o''_i \\ |d_{Y_2}(\hat{\gamma}'_i(t), \hat{\gamma}'_i(s) - |t - s|| &< o'_i + o''_i \\ |d_{Y_2}(\hat{\ell}_i(t), \hat{\gamma}'_i(s)) - (t^2 + s^2)^{\frac{1}{2}}| &< o'_i + o''_i \end{aligned}$$

So both $\hat{\ell}_i$ and $\hat{\gamma}'_i$ converge to independent lines in Y_2 ; thus by the splitting theorem we have that Y_2 splits off an \mathbb{R}^2 . Therefore we can repeat the above arguments, and since N is finite, the process ends after a finite number of steps, ending with a limit Y_k isometric to \mathbb{R}^k for some finite k . \square

3.2. Proof of Theorem 1. The main tool to distinguish between Lie groups and non-Lie groups is the following famous result.

Theorem 3.3 (Gleason-Yamabe). *Let G be a topological group. Suppose that G is not a Lie Group. Then, for every neighbourhood U of the unit in G , there exists a nontrivial subgroup of G contained in U .*

We will proceed by contradiction. Suppose that the isometry group $\text{Iso}(X)$ is not a Lie group. Fix a point $p_0 \in X$ and consider the closed ball $cl(B_{p_0}(1, X))$. For $g \in \text{Iso}(X)$ and $x \in X$, define the displacement function of g as $\delta_g(x, X) = d_X(x, gx)$.

Using Theorem 3.3, we have compact subgroups $K_i \subset \text{Iso}(X)$, $i = 1, \dots$, such that if

$$\delta_i := \max\{\delta_g(x, X) \mid x \in cl(B_{p_0}(1, X)), g \in K_i\},$$

then $\delta_i \rightarrow 0$.

Lemma 3.4. $\delta_i > 0$ for all $i \in \mathbb{N}$.

Proof. Since (X, d_X, \mathbf{m}) is a $RCD^*(K, N)$ -space it is also a $RCD(K, \infty)$ -space and by [12] and [22] these spaces are essentially non-branching.

Now suppose there is some $g \in K_i$ such that $g \equiv id$ on $cl(B_{p_0}(1, X))$ and $g \neq id$ on $X - cl(B_{p_0}(1, X))$.

Take $\mu_0, \mu_1 \in \mathbb{P}_2(X)$ such that $\mu_0 \ll \mathbf{m}$ and is concentrated on $B_{p_0}(\frac{1}{2}, X)$. For μ_1 we ask that both μ_1 and $g_{\#}\mu_1$ are concentrated on disjoint subsets of $X - cl(B_{p_0}(1, X))$.

Let γ be a geodesic in X used in the unique optimal transport between μ_0 and μ_1 . Observe that $g_{\#}\mu_0 = \mu_0$, so we have that $g_{\#}\gamma$ is a geodesic used in the unique optimal transport between μ_0 and $g_{\#}\mu_1$. The hypothesis on g imply that γ is a geodesic that bifurcates after it leaves $cl(B_{p_0}(1, X))$.

This implies that the aforementioned transports are concentrated on a branching set of geodesics, so we have a contradiction. \square

Take $p_i \in cl(B_{p_0}(1, X))$, $g_i \in K_i$ such that $\delta_i := \delta_{g_i}(p_i, X)$. Put $r_i := 1/\delta_i$. Then by Theorem 3.2 there exists a sequence (x_i) , $x_i \rightarrow p_0$, such that $(X, d_{r_i X}, \mathbf{m}_i, x_i)$ converges to an Euclidean space $(\mathbb{R}^m, d_E, \mathcal{L}^m, 0)$, $1 \leq m \leq N$, and $m = m((x_i))$. We will abbreviate $(X, d_{r_i X}, \mathbf{m}_i, x_i)$ as $(r_i X, x_i)$.

We can use Proposition 3.6 of [9] to obtain that $(r_i X, K_i, x_i)$ converges to $(\mathbb{R}^m, K, 0)$ in the pointed equivariant Gromov-Hausdorff distance for some group K .

Lemma 3.5. K is trivial.

Proof. We start by showing that K is compact. Since $x_i \rightarrow p_0$, $x_i \in cl(B_{p_0}(1, X))$ for i large, so

$$\delta_g(x_i, r_i X) \leq \delta_{g_i}(p_i, r_i X) = \delta_i r_i = 1$$

for every $g \in K_i$. This shows that K is compact.

Now, if K has a non-trivial element g there exists a point $y_0 \in \mathbb{R}^m$, such that $\delta_g(y_0, \mathbb{R}^m) = 2$. Take sequences $y_i \rightarrow y_0$, $g_i \rightarrow g$ in $r_i X$, K_i respectively; if o_i is the pointed equivariant Gromov-Hausdorff distance between $r_i X$ and \mathbb{R}^m , then

$$|\delta_{g_i}(y_i, r_i X) - 2| < o_i, \quad o_i \rightarrow 0,$$

and

$$\delta_{g_i}(y_i, X) > \frac{2 - o_i}{r_i} > \delta_i$$

for i large enough. On the other hand, if f_i is an o_i -Gromov-Hausdorff approximation between $(r_i X, x_i)$ and $(\mathbb{R}^m, 0)$, we have

$$|r_i d_X(x_i, y_i) - d_{\mathbb{R}^m}(0, f_i(y_i))| < o_i \implies d_X(x_i, y_i) \leq \frac{o_i + \|f_i(y_i)\|}{r_i} \leq \frac{2o_i + \|y_0\|}{r_i} \rightarrow 0.$$

Therefore for large enough i , $y_i \in cl(B_{p_0}(1, X))$, which contradicts the choice of δ_i . \square

Next we aim to repeat the argument for $r_i X$. For $B'_i := cl(B_{x_i}(1, r_i X))$, put

$$\delta'_i := \max \{ \delta_g(x, r_i X) \mid g \in K_i, x \in B'_i \},$$

and take $p'_i \in B'_i$ and $g'_i \in K_i$ such that $\delta'_i = \delta_{g'_i}(p'_i, r_i X)$. The previous lemma implies that $\delta'_i \rightarrow 0$. We define $r'_i := \frac{1}{\delta'_i} \rightarrow \infty$, and $s_i := r_i r'_i$.

Lemma 3.6. *There exists a sequence (q'_i) in X with $r_i d_X(p'_i, q'_i) \rightarrow 0$, such that, after passing to a subsequence, $(s_i X, q'_i)$ converges to an Euclidean space $(\mathbb{R}^{m'}, d_E, \mathcal{L}^{m'}, 0)$ where $m' = m'((q'_i))$ and $1 \leq m' \leq N$.*

Proof. Since $\text{supp}(\mathbf{m}) = X$ then $\mathbf{m}_i(B'_i) > 0$; by Corollary 1.2 of [18], there exist $q'_i \in B'_i$ such that $d_{r_i X}(p'_i, q'_i) = \epsilon_i \rightarrow 0$, the tangent at q'_i is unique, Euclidean and of dimension $1 \leq m' \leq N$. Notice that we can choose the sequence (q'_i) in such a way that the corresponding tangents have the same dimension.

We construct the required subsequence as follows: for $n, \alpha, \beta \in \mathbb{N}$, define

$$I_n(\alpha, \beta) := \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid \alpha < i, \beta < j, d_{GH}((X, d_{s_i X}, \mathbf{m}_i, q'_i), (\mathbb{R}^{m'}, d_E, \mathcal{L}^{m'}, 0)) < \frac{1}{n}\}.$$

Using the lexicographic order, we define $(i_k, j_k) := \min I_k(i_{k-1}, j_{k-1})$, where $(i_1, j_1) := \min I_1(1, 1)$. It is clear now that $(s_{i_k} X, q'_{j_k})$ is the desired subsequence. \square

Thanks to this lemma we can assume that $(X, d_{s_i X}, \mathbf{m}_i, q'_i) \rightarrow (\mathbb{R}^{m'}, d_E, \mathcal{L}^{m'}, 0)$ and using [9] there exists $K' \subset Iso(\mathbb{R}^{m'})$ such that $(s_i X, K_i, q'_i) \rightarrow (\mathbb{R}^{m'}, K', 0)$ in the pointed equivariant Gromov-Hausdorff distance. Note that K' is compact and that $\delta_{g'_i}(p'_i, s_i X) = 1$ for all i , so it is also nontrivial.

Let $g' \in K'$ be the limit of g'_i ; since it is not the identity, we can find $z_0 \in \mathbb{R}^{m'}$ such that $\delta_{g'}(z_0, \mathbb{R}^{m'}) = 2$. We can then find $z_i \in s_i X$ such that $z_i \rightarrow z_0$ in the Gromov-Hausdorff limit. We would like to guarantee that we can take $z_i \in B'_i$. Let $\theta_i := d_{GH}((X, d_{s_i X}, \mathbf{m}_i, q'_i), (\mathbb{R}^{m'}, d_E, \mathcal{L}^{m'}, 0))$.

Next, we look at the sequence $\{s_i d_X(q'_i, \partial B'_i)\}$, and divide the proof in two parts depending on whether $s_i d_X(q'_i, \partial B'_i)$ is or not bounded above.

In the first case, we have that

$$|s_i d_X(z_i, q'_i) - d_{\mathbb{R}^m}(z_0, 0)| \leq \theta_i,$$

thus $s_i d_X(z_i, q'_i)$ remains bounded, and therefore for i sufficiently large, $z_i \in B'_i$. Since $\delta_{g'_i}(z_i, s_i X) \rightarrow 2$, we get that

$$\delta_{g'_i}(z_i, r_i X) > \frac{2 - \theta_i}{r'_i} > \delta'_i,$$

which contradicts the definition of δ'_i .

Suppose now that $s_i d_X(q'_i, \partial B'_i)$ is bounded above; by passing to a subsequence, we can assume that $s_i d_X(q'_i, \partial B'_i) \rightarrow \limsup_{i \rightarrow \infty} s_i d_X(q'_i, \partial B'_i) < \infty$.

Now let v_i be such that $d_X(q'_i, \partial B'_i) = d_X(q'_i, v_i)$. We then take u_i to be a point in a minimal geodesic between x_i and v_i such that $s_i d_X(u_i, v_i) = 1/(2\theta_i)$. We denote by w_i the midpoint between u_i and v_i .

Lemma 3.7. *Consider $\tilde{B}_i := cl(B_{w_i}(1/4\theta, s_i X))$; then the sequence (\tilde{B}_i, v_i) Gromov-Hausdorff converges to a half space $(H, v) \subset \mathbb{R}^{m'}$*

Proof. We are assuming that the sequence $\{s_i d_X(q'_i, v_i)\}$ converges to a real positive number $a := \limsup_{i \rightarrow \infty} s_i d_X(q'_i, \partial B'_i)$. Now let $f_i : B_{q'_i}(1/\theta_i, s_i X) \rightarrow B_0(1/\theta_i, \mathbb{R}^{m'})$ be a θ_i -Gromov-Hausdorff approximation such that $f_i(q'_i) = 0$, and consider in $\mathbb{R}^{m'}$ the closed ball of diameter $d_E(f_i(u_i), f_i(v_i))$ containing both $f_i(u_i)$ and $f_i(v_i)$. We will denote this ball by \hat{B}_i , and its center by \hat{w}_i .

We want to show that the restrictions to \tilde{B}_i of the Gromov-Hausdorff approximations f_i are still Gromov Hausdorff approximations to \hat{B}_i (with maybe different θ_i 's tending to zero).

First of all, it is easy to see that $f_i(\tilde{B}_i) \subset N_{\theta_i}(\hat{B}_i)$; the Lemma will follow once we show that $\hat{B}_i \subset N_{6\theta_i}(f_i(\tilde{B}_i))$.

To continue, let us estimate first the distance $d_E(\hat{w}_i, f_i(w_i))$. Since \hat{w}_i is the middle point between $f_i(u_i)$ and $f_i(v_i)$, we have that at least one of the angles

$$\alpha := \angle \hat{w}_i f_i(u_i) f_i(w_i), \quad \beta := \angle \hat{w}_i f_i(v_i) f_i(w_i)$$

is less than $\pi/2$; without loss of generality let us suppose it is α . Then

$$\begin{aligned} d_E^2(f_i(w_i), \hat{w}_i) &= \\ &= d_E^2(f_i(u_i), f_i(w_i)) + d_E^2(f_i(u_i), \hat{w}_i) - 2d_E(f_i(u_i), f_i(w_i))d_E(f_i(u_i), \hat{w}_i)\cos(\alpha). \end{aligned}$$

We also have that

$$\begin{aligned} |d_E(f_i(u_i), f_i(w_i)) - d_E(f_i(u_i), \hat{w}_i)| &\leq \\ &= |d_E(f_i(u_i), f_i(w_i)) - 1/4\theta_i| + |d_E(f_i(u_i), \hat{w}_i) - 1/4\theta_i| < 2\theta_i. \end{aligned}$$

Since $\cos \alpha > 0$, we conclude that $d_E(f_i(w_i), \hat{w}_i) < 2\theta_i$.

Let $x \in \tilde{B}_i$ be an arbitrary point, so we have that

$$(4) \quad d_E(x, \hat{w}_i) \leq \frac{1}{2}d_E(f_i(u_i), f_i(v_i)) < \theta_i + \frac{1}{4\theta_i}.$$

Since $f_i : B_{q'_i}(1/\theta_i, s_i X) \rightarrow B_0(1/\theta_i, \mathbb{R}^{m'})$ is a θ_i -Gromov-Hausdorff approximation, there exists some $z \in B_{q'_i}(\frac{1}{\theta_i}, s_i X)$ such that $d_E(x, f_i(z)) < \theta_i$. Observe that together with (4) this implies that

$$(5) \quad d_E(\hat{w}_i, f_i(z)) < 2\theta_i + \frac{1}{4\theta_i}.$$

If $d_{s_i X}(w_i, z) \leq 1/4\theta_i$, then $z \in \tilde{B}_i$, and x would be in the θ_i -neighbourhood of $f_i(\tilde{B}_i)$, so let us assume that $d_{s_i X}(w_i, z) > 1/4\theta_i$.

Choose some minimal geodesic between w_i and z , and let z' be a point in its intersection with $\partial\tilde{B}_i$. Observe that

$$d_{s_iX}(z, z') = d_{s_iX}(w_i, z) - d_{s_iX}(w_i, z') = d_{s_iX}(w_i, z) - \frac{1}{4\theta_i}$$

On the other hand,

$$d_{s_iX}(w_i, z) - \frac{1}{4\theta_i} = (d_{s_iX}(w_i, z) - d_E(f_i(w_i), f_i(z))) + (d_E(f_i(w_i), f_i(z)) - 1/4\theta_i)$$

The first term can be estimated as

$$|d_{s_iX}(w_i, z) - d_E(f_i(w_i), f_i(z))| \leq \theta_i$$

since f_i is a θ_i -Hausdorff approximation. For the second, observe that the inequality (5) yields

$$d_E(f_i(w_i), f_i(z)) \leq d_E(f_i(w_i), \hat{w}_i) + d_E(\hat{w}_i, f_i(z)) < 3\theta_i + \frac{1}{4\theta_i};$$

putting everything together, we finally obtain that

$$d_{s_iX}(z, z') < 4\theta_i,$$

and with this $d_E(f_i(z'), x) < 6\theta_i$.

By using an appropriate element of $SO(m')$ we can assume that $f_i(v_i)$ is contained the nonnegative part of an axis in $\mathbb{R}^{m'}$, now we can take the tangent hyperplane to \hat{B}_i at $f_i(v_i)$. This hyperplane is the boundary of a semispace H_i that contains \hat{B}_i . As $i \rightarrow \infty$ it is clear that the corresponding \hat{B}_i is close to H_i , and from this we obtain a limit semispace H . \square

The proof ends now in a similar way as previously: there exists a point \tilde{z} in the interior of H such that $\delta_{g'}(\tilde{z}, \mathbb{R}^{m'}) = 2$, so this implies that there exists a sequence $(\tilde{z}_i) \subset \tilde{B}_i$ that converges to \tilde{z} and such that

$$\delta_{g'_i}(\tilde{z}_i, r_i X) > \frac{2 - \theta'_i}{r'_i} > \delta'_i$$

and we have the contradiction.

4. UPPER BOUNDS ON THE DIMENSION OF THE ISOMETRY GROUP

For $r > 0$ and Γ , a group acting by isometries on a pointed metric space (X, d_X, p) , we define $\Gamma(r) := \{\gamma \in \Gamma \mid \gamma p \in B_p(r)\}$.

Definition 4.1 (Equivariant Gromov-Hausdorff approximation). *Let (X, d_X, p, Γ) and (Y, d_Y, q, Λ) be two metric spaces and Γ, Λ groups that act isometrically on X and Y respectively. An equivariant pointed ϵ -Gromov-Hausdorff approximation is a triple (f, φ, ψ) of functions*

$$f : B_p(1/\epsilon, X) \rightarrow B_q(1/\epsilon, Y), \quad \varphi : \Gamma(1/\epsilon) \rightarrow \Lambda(1/\epsilon), \quad \psi : \Lambda(1/\epsilon) \rightarrow \Gamma(1/\epsilon)$$

such that

- (1) $f(p) = q$;
- (2) the ϵ -neighbourhood of $f(B_p(1/\epsilon, X))$ contains $B_q(1/\epsilon, Y)$;
- (3) if $x, y \in B_p(1/\epsilon)$, then $|d_X(x, y) - d_Y(f(x), f(y))| < \epsilon$;
- (4) if $\gamma \in \Gamma(1/\epsilon)$ and both $x, \gamma x \in B_p(1/\epsilon, X)$, then

$$d_Y(f(\gamma x), \varphi(\gamma)f(x)) < \epsilon;$$

(5) if $\lambda \in \Lambda(1/\epsilon)$ and both $x, \psi(\lambda)x \in B_p(1/\epsilon, X)$, then

$$d_Y(f(\psi(\lambda)x), \lambda f(x)) < \epsilon.$$

As usual, we do not assume that f is continuous or that φ, ψ are group homomorphisms.

Observe that, given a point $p \in X$ and $r > 0$, the isotropy group G_p acts isometrically on $\partial B_p(r, X)$ and therefore $G_p \leq \text{Iso}(\partial B_p(r, X))$. Since $\partial B_p(r, X)$ is compact, its isometry group is compact, so G_p is compact.

In this section, we will bound the dimension of the isometry group of an $RCD^*(K, N)$ -space. Recall the Theorem mentioned in the introduction.

Theorem 2. *Let (X, d, \mathbf{m}) be a $RCD^*(K, N)$ space with $K, N \in \mathbb{R}$ and $N < \infty$, then the Hausdorff dimension of G , the isometry group of X , is bounded above by $\frac{[N]([N]+1)}{2}$.*

Proof. First we remember that by [18] we have a stratification of X into sets \mathcal{R}_j such that $\mathbf{m}(X - \cup_{1 \leq j \leq [N]} \mathcal{R}_j) = 0$. So let us take $p_0 \in \mathcal{R}_m$, and consider its isotropy group G_{p_0} . Let $\{\lambda_i\} \subset \mathbb{R}$ such that $\lambda_i \rightarrow \infty$ when $i \rightarrow \infty$.

Then we can assume that $(X, d_{\lambda_i X}, \mathbf{m}_i, p_0)$ converges in the measured Gromov-Hausdorff topology to $(\mathbb{R}^m, d_E, \mathcal{L}^m, 0)$. In particular we have Gromov-Hausdorff convergence in the usual sense and by [9] there exists some compact subgroup of isometries of \mathbb{R}^m , K , fixing 0 and such that $(X, d_{\lambda_i X}, p_0, G_{p_0})$ converges in the equivariant Gromov-Hausdorff topology to $(\mathbb{R}^m, d_E, K, 0)$. Observe that for all $R > 0$, $G_{p_0}(R) = G_{p_0}$.

Claim 4.2. *There is an injective Lie group homomorphism $\phi : G_{p_0} \rightarrow K$.*

Proof of claim. Since the action is proper, G_{p_0} is compact and we can use Theorem 3.1 of [13] and assume that the functions used in the ϵ_i -Gromov-Hausdorff approximations $\varphi_i : G_{p_0} \rightarrow K$ are Lie group homomorphisms. We now prove that for large enough i 's, these homomorphisms are injective.

Let $I \in \mathbb{N}$ so that $(f_{\epsilon_i}, \varphi_{\epsilon_i}, \psi_{\epsilon_i})$ is a ϵ_i -Gromov-Hausdorff approximation. First observe that for all $x \in B_{p_0}(1/\epsilon_i, \lambda_i X)$ and all $\gamma \in G_{p_0}$, $\gamma x \in B_{p_0}(1/\epsilon_i, \lambda_i X)$ since p_0 is fixed by the isotropy group.

Then we have that for $\gamma \in \ker \varphi_{\epsilon_i}$, $d_E(f_{\epsilon_i}(\gamma x), f_{\epsilon_i}(x)) < \epsilon_i$ for every $x \in B_{p_0}(1/\epsilon_i, \lambda_i X)$. With this, it is easy to see that $d_X(x, \gamma x) < \frac{2\epsilon_i}{\lambda_i}$.

By passing to a subsequence, if necessary, we can assume that $\epsilon_j \leq \epsilon_i$ and $\lambda_i \leq \lambda_j$ for $i \leq j$. Consider the functions

$$\|\cdot\|_i : G_{p_0} \rightarrow \mathbb{R}^+, \quad \|g\|_i := \sup\{d_X(x, gx) \mid x \in B_{p_0}(1/\epsilon_i, \lambda_i X)\}.$$

Each $\|\cdot\|_i$ is clearly continuous.

Observe that since $\epsilon_j \leq \epsilon_i$, $\lambda_i \leq \lambda_j$, we have $\lambda_i/\epsilon_i \leq \lambda_j/\epsilon_j$, and there are inclusions among balls $B_{p_0}(1/\epsilon_i, \lambda_i X) \subseteq B_{p_0}(1/\epsilon_j, \lambda_j X)$; this implies that for every $g \in \ker \phi_j$, $\|g\|_i \leq \frac{2\epsilon_i}{\lambda_i}$.

The open sets $\|\cdot\|_i^{-1}([0, \frac{2\epsilon_i}{\lambda_i}))$ form a basis of neighbourhoods of the identity in G_{p_0} ; since G_{p_0} is a Lie group, $\ker \phi_j$ becomes trivial for large j . \square

We can bound the dimension of G_{p_0} by $\frac{[N]([N]-1)}{2}$. Now consider the *Myers-Steenrod map*

$$F : G \rightarrow X, \quad F(g) := gp_0.$$

It is easy to see that it is continuous. The quotient map $\pi : G \rightarrow G/G_{p_0}$ is a fibration, thus we can find a compact subset V of G/G_{p_0} and a continuous section $s : V \rightarrow G$ of π passing through the identity. The map $\tilde{F} = F \circ s$ is injective and since V is compact, it is a topological embedding. Then the dimension of G/G_{p_0} is bounded above by $[N]$.

With this we conclude that

$$\dim G = \dim G/G_{p_0} + \dim G_{p_0} \leq \frac{[N]([N] + 1)}{2}$$

as desired. \square

5. RIGIDITY

In this section, we study what $RCD^*(K, N)$ -spaces have isometry group of maximal dimension. We start by recalling the statement of Theorem 3, that affirms that the answer does not differ from the Riemannian case.

Theorem 3. *Let (X, d, \mathbf{m}) be a connected $RCD^*(K, N)$ space of topological dimension $[N]$. If the dimension of the isometry group of X is $[N]([N] + 1)/2$, then X is isometric to one of the following space forms:*

- (1) *An $[N]$ -dimensional Euclidean space $\mathbb{R}^{[N]}$.*
- (2) *An $[N]$ -dimensional sphere $\mathbb{S}^{[N]}$.*
- (3) *An $[N]$ -dimensional projective space $\mathbb{RP}^{[N]}$.*
- (4) *An $[N]$ -dimensional simply connected hyperbolic space $\mathbb{H}^{[N]}$.*

Proof. For the Lie group action of $G = \text{Iso}(X)$ on X , we have that

$$\dim G/G_p + \dim G_p = \dim G,$$

where G_p is any isotropy group. Since the orbit G/G_p embeds topologically in X , we have that $\dim G/G_p \leq \dim X$, and then

$$\frac{[N]([N] + 1)}{2} = \dim G \leq \dim X + \dim G_p \leq [N] + \dim G_p.$$

This yields

$$(6) \quad \dim G_p \geq \frac{[N]([N] - 1)}{2},$$

Choose some point $p_0 \in \mathcal{R}_{[N]}$, and consider a sequence $\lambda_i \rightarrow \infty$ such that $(X, d_i, \mathbf{m}_i, p_0)$ converges in the measured Gromov-Hausdorff topology to $(\mathbb{R}^{[N]}, d_E, \mathcal{L}^{[N]}, 0)$. By [9], there exists a closed subgroup H in $\text{Iso}(\mathbb{R}^{[N]})$ such that the sequence (X, d_i, p_0, G) converges equivariantly to $(\mathbb{R}^{[N]}, d_E, 0, H)$.

On the other hand, consider the same sequence with the restricted group (X, d_i, p_0, G_{p_0}) , where G_{p_0} is the isotropy subgroup of p_0 (or its connected component through the identity, if G_{p_0} was not originally connected). It is clear that this sequence converges as before, but with H replaced by the subgroup of H fixing the origin; we denote this latter group by H_0 . We can once again use Claim 4.2 to conclude that there are injective homomorphisms $\phi_j : G_{p_0} \rightarrow H_0$. Combining this with (6), we get that

$$\dim G_p = \frac{[N]([N] - 1)}{2}, \quad \dim G/G_p = [N], \quad H_0 = SO([N]).$$

Lemma 5.1. *Let (X, d_i, p_0, G) converge equivariantly to $(\mathbb{R}^{[N]}, d_E, 0, H)$ as above. Then H acts transitively on $\mathbb{R}^{[N]}$.*

Proof. From the above, we know that the isotropy group H_0 agrees with $SO([N])$. Let us consider a 1-parameter subgroup $\sigma : \mathbb{R} \rightarrow G$ with $\sigma'(0)$ not tangent to G_{p_0} . Consider the curve $\alpha : \mathbb{R} \rightarrow X$ defined as $\alpha(t) := \sigma(t) \cdot p_0$; then, given $\epsilon > 0$ we can take $\delta := \delta(\epsilon) > 0$ as the largest number such that $c[-\delta, \delta] \subset cl(B_{p_0}(1/\epsilon, X))$. The choice of σ implies that $\alpha(t) \neq p_0$ for some $-t_0 \leq t \leq t_0$.

Let $\varphi_\epsilon : G(1/\epsilon) \rightarrow H(1/\epsilon)$ be the maps used in the equivariant Gromov-Hausdorff approximation; from the choice of δ , $\sigma[-\delta, \delta] \subset G(1/\epsilon)$ and $(\varphi_\epsilon \circ \sigma)[-\delta, \delta] \subset H(1/\epsilon)$. We now define the collection of curves

$$\beta_\epsilon : [-\delta(\epsilon), \delta(\epsilon)] \rightarrow \mathbb{R}^{[N]}, \quad \beta_\epsilon(t) = (\varphi_\epsilon \circ \sigma)(t) \cdot 0$$

Intuitively, the “orbits” $\beta_\epsilon[-\delta, \delta]$ converge to a curve in $\mathbb{R}^{[N]}$ when $\epsilon \rightarrow 0$; this curve will cross any metric sphere centered at 0. We will make this more explicit in what follows.

Since α was not contained in G_{p_0} , there is some $T > 0$ such that $d_X(p_0, \alpha(t)) > 0$ for every $0 < t \leq T$. After rescaling by the λ_i ’s, we obtain that for fixed t as before, $d_i(p_0, \alpha(t)) \rightarrow \infty$ as $i \rightarrow \infty$. By continuity of α , this implies that, given any $r > 0$, there is some positive integer I_0 large enough such that for every $i \geq I_0$, there is some t_i with $d_i(p_0, \alpha(t_i)) = r$.

Observe that, from the definition of α and β_ϵ , we have that

$$d_E(\beta_{\epsilon_i}(t), f_{\epsilon_i}(\alpha(t))) \leq \epsilon_i.$$

Using that we can take each Hausdorff approximation f_{ϵ_i} with $d_E(0, f_{\epsilon_i}(p_0)) \leq \epsilon_i$, the triangle inequality gives us

$$d_E(0, \beta_{\epsilon_i}(t)) \leq \epsilon_i + d_E(f_{\epsilon_i}(p_0), f_{\epsilon_i}(\alpha(t))) + d_E(f_{\epsilon_i}(\alpha(t)), \beta_{\epsilon_i}(t)) \leq 3\epsilon_i + d_{X_i}(p_0, \alpha(t))$$

A similar argument yields the inequality

$$d_{X_i}(p_0, \alpha(t)) \leq d_E(0, \beta_{\epsilon_i}(t)) + 3\epsilon_i$$

Choosing now t_i ’s with $d_i(p_0, \alpha(t_i)) = r$ and passing to infinity, we obtain some point in the H -orbit of 0 at distance r from 0. Since the isotropy group H_0 acted transitively on the unit sphere, we get that H acts transitively on \mathbb{R} , as we wanted to prove. \square

Our next lemma shows that the action of G on X is transitive. This will be a consequence from the transitivity of H on the limit space of the (X, d_i) ’s, but the actual proof is a little more involved.

Lemma 5.2. *If X is a connected $RCD^*(K, N)$ -space with isometry group G of maximal dimension, then G acts transitively on X .*

Proof. The idea of the proof is to show that there is some orbit $G(p)$ in X with non empty interior; since the action of the group is transitive on any of its orbits, $G(p)$ will be open; since it is also closed because G is $\text{Iso}(X)$ with the compact-open topology, the connectedness of X will give the result.

We will now prove that the orbit $G(p_0)$ has positive \mathfrak{m} -measure. Let $\epsilon_i > 0$ be the measured equivariant Gromov-Hausdorff distances between (X, d_i) and $(\mathbb{R}^{[N]}, d_E)$, and consider $G(p_0) \cap \bar{B}_{p_0}(1/\epsilon_i, X)$. For an ϵ_i -Gromov-Hausdorff approximation f_{ϵ_i} , we have that

$$f_{\epsilon_i}(G(p_0) \cap cl(B_{p_0}(1/2\epsilon_i, X, d_i))) \subset cl(B_0(1/\epsilon_i, \mathbb{R}^{[N]}))$$

whenever i is large enough.

For any $y \in cl(B_0(1/2\epsilon_i, \mathbb{R}^{[N]}))$, there exists some point $x \in cl(B_{p_0}(1/2\epsilon_i, X))$ such that $d_E(y, f_{\epsilon_i}(x)) < \epsilon_i$. Since H acted transitively on $\mathbb{R}^{[N]}$, there exists some element $h_i \in H(1/\epsilon_i)$ such that $h_i \cdot y = 0$; this yields

$$d_E(f_{\epsilon_i}(x), h_i^{-1} \cdot 0) = d_E(h_i f_{\epsilon_i}(x), 0) < \epsilon_i.$$

For the map $\psi_{\epsilon_i} : H(1/\epsilon_i) \rightarrow G(1/\epsilon_i)$ appearing in the definition of Gromov-Hausdorff convergence, we get

$$\psi_{\epsilon_i}(h_i^{-1}) \in G(1/\epsilon_i), \quad d_E(f_{\epsilon_i}(\psi_{\epsilon_i}(h_i^{-1})p_0), h_i^{-1}0) < \epsilon_i.$$

From the triangle inequality, we obtain

$$d_E(f_{\epsilon_i}(\psi_{\epsilon_i}(h_i^{-1})p_0), y) \leq d_E(f_{\epsilon_i}(\psi_{\epsilon_i}(h_i^{-1})p_0), h_i^{-1} \cdot 0) + d_E(h_i^{-1} \cdot 0, f_{\epsilon_i}(x)) + d_E(f_{\epsilon_i}(x), y) < 3\epsilon_i.$$

This implies that for sufficiently small ϵ_i , the piece of the orbit $G(p_0) \cap cl(B_{p_0}(1/2\epsilon_i, X))$ is sufficiently close to the Euclidean ball $B(0, 1/2\epsilon_i)$, and since $f_{\epsilon_i \#} \mathbf{m}$ converges to $\mathcal{L}^{[N]}$, then it has positive \mathbf{m} -measure.

Now fix some $\epsilon'_i < \epsilon_i/2$, and consider the closure of the ϵ'_i -neighbourhood $N := cl(N_{\epsilon'_i}(G(p_0)) \cap cl(B_{p_0}(1/\epsilon_i, X)))$; N admits a decomposition into orbits of the form $N = \cup_q G(q) \cap cl(B_{p_0}(1/\epsilon_i, X))$ where $q \in N$; as in the previous argument, it follows that for all q the corresponding piece of the orbit through q has positive \mathbf{m} -measure. Since $(N, d|_N, \frac{1}{\mathbf{m}(N)} \mathbf{m})$ is also Polish, there is only a countable number of orbits in N . From Baire's Category theorem, we have that one of them must have non-empty interior; without loss of generality we can assume that it is $G(p_0)$. As mentioned at the beginning of the proof, the hypothesis that X is connected implies that $X = G(p_0)$. \square

Now we have that G acts transitively on X , X is homeomorphic to G/G_{p_0} and therefore it is locally contractible. Theorem 3 in [3] characterizes X as isometric to the quotient of G by G_{p_0} endowed with a Carnot-Carathéodory-Finsler metric d_{cc} that corresponds to a k -dimensional tangent distribution Δ , and a norm \mathcal{F} defined over it. In order to ensure that the metric is actually Riemannian, we now look at the tangent spaces corresponding to these structures.

On the one hand, we have that at p_0 , X has a unique tangent space of the form $(\mathbb{R}^{[N]}, d_E, \mathcal{L}^{[N]}, 0)$ obtained by means of measured Gromov-Hausdorff convergence; note that the Hausdorff dimension of this space is $[N]$.

On the other hand the tangent spaces of spaces of the form $(G/G_{p_0}, d_{cc})$ are Carnot groups whose Hausdorff dimension is, by [19], greater or equal than $[N]$, with equality being achieved if and only if the dimension of Δ is exactly $[N]$. Note that this tangent space is obtained by using Gromov-Hausdorff convergence without any requirement on a measure.

By uniqueness of the tangent spaces we have that these must be isometric, so they must have the same Hausdorff dimension. It follows then that the distribution Δ has dimension $[N]$. From this we have by Theorem 7 of [3] that X is actually a Finsler manifold; since it is also an $RCD^*(K, N)$ -space, it must have quadratic Cheeger energy, and therefore be in fact a Riemannian manifold [1]. Theorem follows now from Theorem 3.1 of [14], that characterizes Riemannian manifolds with maximal isometry groups.

Finally note that by [7] the measure \mathbf{n} that we give to the resulting Riemannian manifold and the $[N]$ -dimensional Hausdorff measure are mutually absolutely continuous. \square

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